

ARITHMETIC PROPERTIES OF EIGENVALUES OF GENERALIZED HARPER OPERATORS ON GRAPHS

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ABSTRACT. Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in \mathbb{C} . A discrete group G is said to have the σ -multiplier algebraic eigenvalue property, if for every matrix $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$, regarded as an operator on $l^2(G)^d$, the eigenvalues of A are algebraic numbers, where $\sigma \in Z^2(G, \mathcal{U}(\overline{\mathbb{Q}}))$ is an algebraic multiplier, and $\mathcal{U}(\overline{\mathbb{Q}})$ denotes the unitary elements of $\overline{\mathbb{Q}}$. Such operators include the Harper operator and the discrete magnetic Laplacian that occur in solid state physics. We prove that any finitely generated amenable, free or surface group has this property for any algebraic multiplier σ . In the special case when σ is rational ($\sigma^n=1$ for some positive integer n) this property holds for a larger class of groups \mathcal{K} containing free groups and amenable groups, and closed under taking directed unions and extensions with amenable quotients. Included in the paper are proofs of other spectral properties of such operators.

1. INTRODUCTION

This paper is concerned with number theoretic properties of eigenvalues of self adjoint matrix operators that are associated with weight functions on a graph equipped with a free action of a discrete group. These operators form generalizations of the Harper operator and the discrete magnetic Laplacian (DML) on such graphs, as defined by Sunada in [23].

The Harper operator and DML over the Cayley graph of \mathbb{Z}^2 arise as the Hamiltonian in discrete models of the behaviour of free electrons in the presence of a magnetic field, where the strength of the magnetic field is encoded in the weight function. When the weight function is trivial, the Harper operator and the DML reduce to the Random Walk operator and the discrete Laplacian respectively. The DML is in particular the Hamiltonian in a discrete model of the integer quantum Hall effect (see for example [3]); when the graph is the Cayley graph of a cocompact Fuchsian group, the DML becomes the Hamiltonian in a discrete model of the fractional quantum Hall effect ([6], [7] and [17]). It has also been studied in the context of noncommutative Bloch theory ([16], [21].)

The Harper operator and DML can be thought of as particular examples of weighted sums of twisted right translations by elements of the group; alternatively, they can be regarded as matrices over the group algebra twisted by a 2-cocycle, acting by (twisted) left multiplication. As such, in section 6 we generalize results of [9] to demonstrate that such operators associated with algebraic weight functions have only algebraic eigenvalues whenever the group is in a class

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of groups containing all free groups, finitely generated amenable groups and fundamental groups of closed Riemann surfaces.

When the multiplier associated to the weight function is rational, the algebraicity of eigenvalues extends to groups in a larger class \mathcal{K} , defined in section 8. The class \mathcal{K} contains all free groups, discrete amenable groups, and groups in the Linnell class \mathcal{C} ; in particular it includes cocompact Fuchsian groups and many other non-amenable groups. The algebraic eigenvalue properties derived in this paper can be summarized in the following theorem.

Theorem 1.1 (Corollary 4.5, Theorems 6.1, 6.2, 6.3). *Let σ be an algebraic multiplier for the discrete group G , and let $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ be an operator acting on $l^2(G)^d$ by left multiplication twisted by σ , where $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers. Alternatively, consider A to be a finite sum of magnetic translation operators $\sum_{g \in G} w_g R_g^\sigma$, where $w_g \in M_d(\overline{\mathbb{Q}})$. Then A has only algebraic eigenvalues whenever*

- (1) G is finitely generated amenable, free or a surface group, or
- (2) σ is rational ($\sigma^n = 1$ for some positive integer n) and $G \in \mathcal{K}$, where \mathcal{K} is a class of groups containing free groups and amenable groups, and is closed under taking directed unions and extensions with amenable quotients.

The case when σ is a rational multiplier is established by relating the spectrum of these operators to the spectrum of untwisted operators on a finite covering graph. The property follows from the class \mathcal{K} having the (untwisted) algebraic eigenvalue property (established in section 6, following [9]), and from the fact that \mathcal{K} is closed under taking extensions with cyclic kernel, as demonstrated in section 8. We show in section 4 that these operators with rational weight function have no eigenvalues that are Liouville transcendental whenever the group is residually finite or more generally in a certain large class of groups $\hat{\mathcal{G}}$ containing \mathcal{K} . We also show that there is an upper bound for the number of eigenvalues whenever the group satisfies the Atiyah conjecture.

However, the case when σ is an algebraic multiplier is established in a significantly different manner: in addition to an approximation argument that parallels that of [9], one also has to use new arguments that rely upon the geometry of closed Riemann surfaces.

We also wish to highlight the remarkable computation of Grigorchuk and Żuk [13], that is recalled in Theorem 5.5. The computation explicitly lists the dense set of eigenvalues of the Random Walk operator on the Cayley graph of the lamplighter group — all of these eigenvalues are algebraic numbers, as predicted by results in this paper and in [18], since the lamplighter group is an amenable group.

Section 7 establishes an equality between the von Neumann spectral density function of $A \in M_d(\mathbb{C}(G, \sigma))$ for arbitrary multiplier σ , and the integrated density of states of A with respect to a generalized Følner exhaustion of G , whenever G is a finitely generated amenable group, or a surface group.

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2. MAGNETIC TRANSLATIONS AND THE TWISTED GROUP ALGEBRA

The Harper operator is an example of an operator that can be described as a sum of magnetic right translations. In this section we will offer a brief description of the magnetic translation operators, and observe that finite weighted sums of these magnetic translations are unitarily equivalent to left multiplication by matrices over a twisted group algebra.

Let G be a discrete group and σ be a *multiplier*, that is $\sigma \in Z^2(\Gamma; U(1))$ is a normalized $U(1)$ -valued cocycle, which is a map from $\Gamma \times \Gamma$ to $U(1)$ satisfying

$$\begin{aligned} (1) \quad & \sigma(b, c)\sigma(a, bc) = \sigma(ab, c)\sigma(a, b) \quad \forall a, b, c \in G \\ (2) \quad & \sigma(1, g) = \sigma(g, 1) = 1 \quad \forall g \in G \end{aligned}$$

Consider the Hilbert space $l^2(G)^d$ of square-integrable \mathbb{C}^d -valued functions on G , where $l^2(G) = \{h: G \rightarrow \mathbb{C} : \sum_{g \in G} |h(g)|^2 < \infty\}$. The *right magnetic translations* are then defined by

$$(R_g^\sigma f)(x) = f(xg)\sigma(x, g).$$

Obviously, if σ is a multiplier, so is $\bar{\sigma}$. The right magnetic translations commute with left magnetic translations $L_h^{\bar{\sigma}}$ as follows from (1):

$$(3) \quad R_g^\sigma L_h^{\bar{\sigma}} = L_h^{\bar{\sigma}} R_g^\sigma \quad \forall g, h \in G$$

where

$$(L_h^{\bar{\sigma}} f)(x) = f(h^{-1}x) \overline{\sigma(h, h^{-1}x)}.$$

Consider now self-adjoint operators on $l^2(G)^d$ of the form

$$(4) \quad A^\sigma = \sum_{g \in S} A(g) R_g^\sigma$$

where $A(g)$ is a $d \times d$ complex matrix for each g , and S is a finite subset of G . The self-adjointness condition is equivalent to demanding that the weights $A(g)$ satisfy $A(g)^* = A(g^{-1})\sigma(g, g^{-1})$.

These operators include as a special case the Harper operator and the DML on the Cayley graph of G , where S is the generating set and $A(g)$ is identically 1 for g in S . For the Harper operator of Sunada [23] on a graph with finite fundamental domain under the free action of the group G , one can construct an operator of the form (4) which is unitarily equivalent to the Harper operator, after identifying scalar valued functions on the graph with \mathbb{C}^n -valued functions on the group, where n is the size of the fundamental domain.

Note that the operators of the form (4) are given explicitly by the formula

$$(A^\sigma f)(x) = \sum_{g \in G} A(x^{-1}g)\sigma(x, x^{-1}g)f(g).$$

For a given multiplier σ taking values in $\mathcal{U}(K) = K \cap U(1)$ for some subfield K of the complex numbers, one can also construct the twisted group algebra $K(G, \sigma)$ and examine $d \times d$ matrices $B^\sigma \in M_d(K(G, \sigma))$ acting on $l^2(G)^d$. Elements of $K(G, \sigma)$ are finite sums $\sum a_g g$, $a_g \in K$ with multiplication given by

$$\left(\sum a_g g\right) \cdot \left(\sum b_h h\right) = \sum_{gh=k} a_g b_h \sigma(g, h) k.$$

The action of B^σ on a \mathbb{C}^d -valued function f is then given by this multiplication,

$$(B^\sigma f)(x) = \sum_{h \in G} B(xh^{-1})\sigma(xh^{-1}, h)f(h)$$

where $B(g)$ denotes the $d \times d$ matrix over K whose elements are the coefficients of g in the elements of B . A straightforward computation shows that

$$(5) \quad B^\sigma = \sum_{g \in S} B(g) L_g^\sigma,$$

where S is a finite subset of G .

The left and right twisted translations are unitarily equivalent via the map U^σ ,

$$(U^\sigma f)(x) = \sigma(x, x^{-1}) f(x^{-1}), \\ U^\sigma L_g^\sigma = R_g^\sigma U^\sigma.$$

As such, operators of the form (4) and (5) will be unitarily equivalent if the coefficient matrices satisfy $A(g) = B(g)$ for all $g \in G$. Hereafter we will therefore concentrate on the latter picture, noting that the results apply equally well to the case of operators described as weighted sums of right magnetic translations.

By virtue of (3), the operators B^σ of the form (5) belong to the commutant $B(l^2(G)^d)^{G, \sigma}$ of the set of magnetic translations $\{R_g^\sigma \mid g \in G\}$; the weak closure of the set of operators of the form (5) is actually equal to this commutant as the theorem below shows.

The theorem itself is folklore, but we were not able to find the proof in the literature. In the special case of $G = \mathbb{Z}^2$, the details are spelt out in [21]. We will give a self-contained account, adapting the proof for the case of trivial multiplier.

Theorem 2.1 (Commutant theorem). *The commutant of the right σ -translations on $l^2(G)$ is the von Neumann algebra generated by left $\bar{\sigma}$ -translations on $l^2(G)$.*

Similarly, the commutant of the left σ -translations on $l^2(G)$ is the von Neumann algebra generated by right $\bar{\sigma}$ -translations on $l^2(G)$.

Proof. We present a proof of the second statement: the proof of the first statement is analogous.

Let $W_{L, \sigma}$ be the von Neumann algebra generated by the set $S_{L, \sigma} = \{L_g^\sigma \mid g \in G\}$ of left σ -translations, and $W_{R, \bar{\sigma}}$ be the von Neumann algebra generated by the set $S_{R, \bar{\sigma}} = \{R_g^{\bar{\sigma}} \mid g \in G\}$ of right $\bar{\sigma}$ -translations. We proceed by showing that $S'_{L, \sigma} = S''_{R, \bar{\sigma}}$ (denoting the commutant by $'$) and then show that $S''_{R, \bar{\sigma}} = W_{R, \bar{\sigma}}$.

An operator $C \in B(l^2(G))$ is determined by its components $C_{a,b} = (C\delta_b, \delta_a) = (C\delta_b)(a)$ for $a, b \in G$. Suppose $C \in S'_{R, \bar{\sigma}}$. In terms of components, one has that $(R_g^{\bar{\sigma}})_{a,b} = \delta_b(ag)\overline{\sigma(a,g)} = \delta_a(bg^{-1})\overline{\sigma(bg^{-1}, g)}$, giving $(CR_g^{\bar{\sigma}})_{a,b} = C_{a, bg^{-1}}\overline{\sigma(bg^{-1}, g)}$ and $(R_g^{\bar{\sigma}}C)_{a,b} = \overline{\sigma(a, g)}C_{ag, b}$. C commutes with $R_g^{\bar{\sigma}}$ for all g , and so substituting bg for b gives

$$C \in S'_{R, \bar{\sigma}} \implies C_{a,b} = \overline{\sigma(a, g)} C_{ag, bg} \sigma(b, g) \quad \forall a, b, g \in G.$$

Similarly for $D \in S'_{L, \sigma}$, noting that $(L_g^\sigma)_{a,b} = \delta_b(g^{-1}a)\sigma(g, g^{-1}a) = \delta_a(gb)\sigma(g, b)$, we have $(DL_g^\sigma)_{a,b} = D_{a, gb}\sigma(g, b)$ and $(L_g^\sigma D)_{a,b} = \sigma(g, g^{-1}a)D_{g^{-1}a, b}$, which after substituting ga for a gives

$$D \in S'_{L, \sigma} \implies D_{a,b} = \overline{\sigma(g, a)} D_{ga, gb} \sigma(g, b) \quad \forall a, b, g \in G.$$

Consider the product CD for $C \in S'_{R,\bar{\sigma}}$ and $D \in S'_{L,\sigma}$. In terms of components,

$$\begin{aligned} (CD)_{a,b} &= \sum_{g \in G} C_{a,g} D_{g,b} = \sum_{g \in G} \phi(a, g^{-1}, b) C_{a(g^{-1}b), b} D_{a, (ag^{-1})b} \\ &= \sum_{h \in G} \phi(a, a^{-1}hb^{-1}, b) D_{a,h} C_{h,b}, \end{aligned}$$

where $\phi(a, g^{-1}, b) = \overline{\sigma(a, g^{-1}b)} \sigma(g, g^{-1}b) \overline{\sigma(ag^{-1}, g)} \sigma(ag^{-1}, b)$. However we can reduce the expression for ϕ by applying the cocycle identities to show that it is in fact identically equal to 1:

$$\begin{aligned} \phi(a, k, b) &= \sigma(k^{-1}, kb) \left(\sigma(ak, b) \overline{\sigma(a, kb)} \right) \overline{\sigma(ak, k^{-1})} \\ &= \sigma(k^{-1}, kb) \sigma(k, b) \overline{\sigma(a, k) \sigma(ak, k^{-1})} \\ &= \sigma(k^{-1}, k) \overline{\sigma(k, k^{-1})} \\ &= 1 \quad \forall a, k, b \in G. \end{aligned}$$

So $(CD)_{a,b} = (DC)_{a,b}$ for all $a, b \in G$, demonstrating that operators in $S'_{L,\sigma}$ commute with those in $S'_{R,\bar{\sigma}}$.

This gives the inclusion $S'_{L,\sigma} \subset S''_{R,\bar{\sigma}}$. The left σ -translations and right $\bar{\sigma}$ -translations commute, so we also have that $S_{L,\sigma} \subset S'_{R,\bar{\sigma}}$ and thus $S''_{R,\bar{\sigma}} \subset S'_{L,\sigma}$. Therefore $S'_{L,\sigma} = S''_{R,\bar{\sigma}}$.

A calculation shows that the adjoint of $R_g^{\bar{\sigma}}$ is given by $(R_g^{\bar{\sigma}})^* = \sigma(g, g^{-1}) R_{g^{-1}}^{\bar{\sigma}}$, and so operators that commute with the right $\bar{\sigma}$ -translations must commute with their adjoints as well. So $S'_{R,\bar{\sigma}} = S'^{*}_{R,\bar{\sigma}}$, writing S^* for the set of adjoints of elements of S .

By the von Neumann double commutant theorem, the algebra generated by a set S is given by $(S \cup S^*)''$. So $W_{R,\bar{\sigma}} = (S_{R,\bar{\sigma}} \cup S'^{*}_{R,\bar{\sigma}})'' = S''_{R,\bar{\sigma}} = S'_{L,\sigma}$. \square

We set the notation $W_L^*(G, \sigma) = W_{L,\sigma} = S'_{R,\bar{\sigma}}$ for the left twisted group von Neumann algebra and $W_R^*(G, \sigma) = W_{R,\sigma} = S'_{L,\bar{\sigma}}$ for the right twisted group von Neumann algebra.

The following is a corollary of the theorem.

Corollary 2.2. *The commutant of the right σ -translations on $l^2(G)^d$ is the von Neumann algebra $W_L^*(G, \bar{\sigma}) \otimes M_d(\mathbb{C})$.*

Similarly, the commutant of the left σ -translations on $l^2(G)^d$ is the von Neumann algebra $W_R^(G, \bar{\sigma}) \otimes M_d(\mathbb{C})$.*

Theorem 2.3 (Existence of trace). *There is a canonical faithful, finite and normal trace on the twisted group von Neumann algebras $W_L^*(G, \sigma)$ and $W_R^*(G, \bar{\sigma})$ which is given by*

$$(6) \quad \text{tr}_{G,\sigma}(A) = (A\delta_e, \delta_e).$$

This trace is weakly continuous and can also be written as

$$(7) \quad \text{tr}_{G,\sigma}(A) = (A\delta_g, \delta_g), \quad g \in G.$$

Proof. It is clear that $\text{tr}_{G,\sigma}$ is linear, finite and weakly continuous (hence normal). Now if $A \in W_L^*(G, \sigma)$, then

$$(8) \quad (A\delta_g, \delta_g) = A_{g,g} = \sigma(g, h) A_{gh,gh} \overline{\sigma(g, h)} = A_{gh,gh} = (A\delta_{gh}, \delta_{gh})$$

for all $h \in G$. In particular, every diagonal entry of the matrix of A is equal to $\text{tr}_{G,\sigma}(A)$.

If A is a self-adjoint operator in $W_L^*(G, \sigma)$ such that $\text{tr}_{G,\sigma}(A) = 0$, then $(A\delta_g, \delta_g) = 0$ for all $g \in G$. But then due to the Cauchy-Schwarz inequality $|(Af_1, f_2)|^2 \leq (Af_1, f_1)(Af_2, f_2)$, $f_1, f_2 \in l^2(G)$, we deduce that $(A\delta_g, \delta_h) = 0$ for all $g, h \in G$, which implies that $A = 0$. Therefore $\text{tr}_{G,\sigma}$ is faithful.

It remains to prove that $\text{tr}_{G,\sigma}$ is a trace. That is,

$$(9) \quad \text{tr}_{G,\sigma}(AB) = \text{tr}_{G,\sigma}(BA), \quad A, B \in W_L^*(G, \sigma)$$

Since $\text{tr}_{G,\sigma}$ is linear and weakly continuous it is sufficient to consider the case when $A = L_g^\sigma$ and $B = L_h^\sigma$ for all $g, h \in G$. We compute,

$$\begin{aligned} \text{tr}_{G,\sigma}(L_g^\sigma L_h^\sigma) &= (L_g^\sigma L_h^\sigma \delta_e, \delta_e) = \sigma(g, h)(L_{gh}^\sigma \delta_e, \delta_e) = \sigma(g, h)(\delta_{gh}, \delta_e) \\ &= \begin{cases} \sigma(g, h) & \text{if } gh = e, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly,

$$\text{tr}_{G,\sigma}(L_h^\sigma L_g^\sigma) = \begin{cases} \sigma(h, g) & \text{if } hg = e, \\ 0 & \text{otherwise.} \end{cases}$$

By the cocycle identity (1) with $a = h^{-1}$, $b = h$ and $c = h^{-1}$, we see that

$$\sigma(h, h^{-1}) = \sigma(h^{-1}, h) \quad \forall h \in G.$$

Therefore $\text{tr}_{G,\sigma}(L_g^\sigma L_h^\sigma) = \text{tr}_{G,\sigma}(L_h^\sigma L_g^\sigma)$ for all $g, h \in G$ as required. The argument for $W_R^*(G, \bar{\sigma})$ is identical. \square

Now the matrix algebra $M_d(\mathbb{C})$ has the canonical trace Tr given by the sum of the diagonal coefficients of a matrix. Then the tensor product $\text{tr}_{G,\sigma} \otimes \text{Tr}$ is a trace on $W_L^*(G, \sigma) \otimes M_d(\mathbb{C})$ and on $W_R^*(G, \bar{\sigma}) \otimes M_d(\mathbb{C})$ which we denote by $\text{tr}_{G,\sigma}$ for simplicity.

Corollary 2.4. *There is a canonical faithful, finite and normal trace on the twisted group von Neumann algebras $W_L^*(G, \sigma) \otimes M_d(\mathbb{C})$ and on $W_R^*(G, \bar{\sigma}) \otimes M_d(\mathbb{C})$ which is given by*

$$(10) \quad \text{tr}_{G,\sigma}(A) = \sum_{j=1}^d (A_{j,j} \delta_e, \delta_e).$$

This trace is weakly continuous and can also be written as

$$(11) \quad \text{tr}_{G,\sigma}(A) = \sum_{j=1}^d (A_{j,j} \delta_g, \delta_g), \quad g \in G.$$

In the corollary above, we interpret the elements of the tensor product $W_L^*(G, \sigma) \otimes M_d(\mathbb{C}) = M_d(W_L^*(G, \sigma))$ as $d \times d$ matrices with entries in $W_L^*(G, \sigma)$ etc.

Suppose that \mathcal{A} is a von Neumann algebra of algebras of operators acting on a Hilbert space \mathcal{H} . A subspace U of \mathcal{H} is termed *affiliated* if the corresponding orthogonal projection P_U onto the closure of U belongs to \mathcal{A} . A necessary and sufficient condition for affiliation is that the subspace

be invariant under the action of operators in the commutant \mathcal{A}' of \mathcal{A} . We will write $U\eta\mathcal{A}$ to indicate that the subspace U is affiliated to the algebra \mathcal{A} .

Given a trace τ on \mathcal{A} , the von Neumann dimension \dim_τ of an affiliated subspace is defined to be the trace of P_U . We will use the following properties of the von Neumann dimension (see for example [22], Section 2.6 Lemma 2, Section 2.26).

Lemma 2.5. *Let \mathcal{H} be a Hilbert space, \mathcal{A} a von Neumann algebra of operators on \mathcal{H} with (normal, faithful and semi-finite) trace τ and von Neumann dimension \dim_τ , and let $L, N \subset \mathcal{H}$ be affiliated subspaces. Then,*

- (1) $\dim_\tau L = 0$ implies $L = \{0\}$,
- (2) $L \subseteq N$ implies $\dim_\tau L \leq \dim_\tau N$,
- (3) if $A \in \mathcal{A}$ is an almost isomorphism of L and N , that is, $\ker A \cap L = \{0\}$ and the set $A(L)$ is dense in N , then $\dim_\tau L = \dim_\tau N$.

The following is an immediate consequence.

Lemma 2.6. *Let \mathcal{H} , \mathcal{A} , τ be as in Lemma 2.5. If L is an affiliated subspace of \mathcal{H} with corresponding projection P_L , then*

$$\dim_\tau L = \dim_\tau \ker A|_L + \dim_\tau \overline{\operatorname{im} A|_L} = \dim_\tau (\ker A \cap L) + \dim_\tau \overline{\operatorname{im} AP_L}.$$

Proof. This follows by noting that A gives an almost isomorphism from the orthogonal complement of its kernel in L to the closure of its image on L . \square

Hereafter we will use $\dim_{G,\sigma}$ to refer to the von Neumann dimension associated with the trace $\operatorname{tr}_{G,\sigma}$ on the algebra $W_L^*(G, \sigma) \otimes M_d(\mathbb{C})$. In the case that σ is trivial, this algebra becomes the von Neumann algebra of G -equivariant operators $B(l^2(G)^d)^G$, and we refer to the trace and dimension by tr_G and \dim_G respectively. To make the notation more suggestive we will also write $W_L^*(G, \sigma) \otimes M_d(\mathbb{C}) = B(l^2(G)^d)^{G,\sigma}$.

Two multipliers σ and σ' in $\mathbb{Z}^2(G, \mathcal{U}(K))$ are *cohomologous*, written $\sigma \sim \sigma'$, if they belong to the same cohomology class in $H^2(G, \mathcal{U}(K))$. It follows that $\sigma \sim \sigma'$ if and only if there exists a map $s : G \rightarrow \mathcal{U}(K)$ such that

$$(12) \quad \sigma(g, f) = s(g)s(h)\overline{s(gh)}\sigma'(g, h) \quad \forall g, h \in G.$$

The map s gives rise to a unitary equivalence between operators in $M_d(K(G, \sigma))$ and $M_d(K(G, \sigma'))$.

Lemma 2.7. *Let σ and σ' be cohomologous multipliers in $\mathbb{Z}^2(G, \mathcal{U}(K))$. Then for every A in $M_d(K(G, \sigma))$ acting on $l^2(G)^d$ there is a canonically determined A' in $M_d(K(G, \sigma'))$ such that A and A' are unitarily equivalent.*

Proof. Let $s : G \rightarrow \mathcal{U}(K)$ be the map as in (12), such that $\sigma(g, h) = s(g)s(h)\overline{s(gh)}\sigma'(g, h)$ for all $x, y \in G$. Writing $A(g) \in M_d(K)$ for the matrix of coefficients of g in A , as in (5), one has

$$\begin{aligned} (Af)(g) &= \sum_{h \in G} A(gh^{-1})f(h)\sigma(gh^{-1}, h) \\ &= \sum_{h \in G} A(gh^{-1})f(h)s(gh^{-1})s(h)s(g)^{-1}\sigma'(gh^{-1}, h). \end{aligned}$$

Let S be the unitary operator on $l^2(G)^d$ given by multiplication by s : $Sf(g) = s(g)f(g)$. Then letting $A'(g) = s(g)A(g)$, one has

$$\begin{aligned} (SAf)(g) &= \sum_{h \in G} A'(gh^{-1})f(h)s(h)\sigma'(gh^{-1}, h). \\ &= (A'Sf)(g) \end{aligned}$$

for all $g \in G$, $f \in l^2(G)^d$. That is, A and A' are unitarily equivalent. \square

It is sometimes convenient to consider only the case when the multiplier satisfies $\sigma(g, g^{-1}) = 1$ for all g in G . The following lemma shows that there is such a multiplier in every cohomology class when the subfield K is algebraically closed.

Lemma 2.8. *Suppose K is an algebraically closed subfield of \mathbb{C} . Then any multiplier $\sigma \in Z^2(G, \mathcal{U}(K))$ is cohomologous to a multiplier σ' such that $\sigma'(g, g^{-1}) = 1$ for all $g \in G$.*

Proof. By the cocycle identity, $\sigma(g, g^{-1}) = \sigma(g^{-1}, g)$ for all $g \in G$. Choose $s : G \rightarrow \mathcal{U}(K)$ such that $s(g) = s(g^{-1})$ and $s(g)^2 = \sigma(g, g^{-1})$, for example by setting $s(g) = e^{i\theta/2}$ when $\sigma(g, g^{-1}) = e^{i\theta}$, for $\theta \in [0, 2\pi)$. The image of s lies in $\mathcal{U}(K)$ due to K being algebraically closed.

Let σ' be the cohomologous multiplier given by s , according to the formula (12). Then

$$\begin{aligned} \sigma'(g, g^{-1}) &= \overline{s(g)s(g^{-1})}s(1)\sigma(g, g^{-1}) \\ &= \overline{\sigma(g, g^{-1})}\sigma(g, g^{-1}) \\ &= 1, \quad \forall g \in G. \end{aligned}$$

\square

3. ALGEBRAIC EIGENVALUE PROPERTY

The algebraic eigenvalue property for groups was introduced in [9]. We recall the definition here, and present a class of groups \mathcal{K} for which the algebraic eigenvalue property holds. We then define a similar property describing the eigenvalues for matrix operators over the twisted group ring, as described in section 2, termed the σ -multiplier algebraic eigenvalue property.

Thus recall the following definition from [9], where $\overline{\mathbb{Q}}$ denotes the set of complex algebraic numbers.

Definition 3.1 (4.1 of [9]). A discrete group G has the *algebraic eigenvalue property*, if for every $d \times d$ matrix $A \in M_d(\overline{\mathbb{Q}}G)$ the eigenvalues of A , acting on $l^2(G)^d$, are algebraic numbers.

Note that operators without point spectrum satisfy the criterion in the vacuous sense.

The trivial group has the algebraic eigenvalue property, since the eigenvalues are the zeros of the characteristic polynomial. The same is true for every finite group. More generally, if G contains a subgroup H of finite index, and H has the algebraic eigenvalue property, then the same is true for G . And if G has the algebraic eigenvalue property and H is a subgroup of G , then H also has the algebraic eigenvalue property.

In section 4 of [9] it was shown that the algebraic eigenvalue property holds for all amenable groups and for all groups in Linnell's class \mathcal{C} , which is the smallest class of groups containing all free groups and which is closed under extensions with elementary amenable quotient and under directed unions. This motivates the definition of the class \mathcal{K} , a larger class which contains these groups, for which the algebraic eigenvalue property can be shown to hold.

Definition 3.2. The class \mathcal{K} is the smallest class of groups containing free groups and amenable groups, which is closed under taking extensions with amenable quotient, and under taking directed unions.

Remark 3.3. It is clear that the class \mathcal{K} contains every discrete amenable group and every group in Linnell's class \mathcal{C} .

Recall that the class of elementary amenable groups is the smallest class of groups containing all cyclic and all finite groups and which is closed under taking group extensions and directed unions. As such then \mathcal{K} is a strictly larger class of groups than \mathcal{C} , as it contains amenable groups which are not elementary amenable, such as the example presented by Grigorchuk in [12].

Remark 3.4. Every subgroup of infinite index in a surface group Γ is a free group. Here Γ is the fundamental group of a compact Riemann surface of genus $g > 1$. This follows from the fact that such groups are fundamental groups of an infinite cover of the base surface and from the general fact that the fundamental group of a noncompact surface is free (see [1] Chapter 1, § 7.44 and § 8.) Since we have the exact sequence

$$1 \rightarrow F \rightarrow \Gamma \rightarrow \mathbb{Z}^{2g} \rightarrow 1$$

where F is a free group by the argument above and the free abelian group \mathbb{Z}^{2g} is an elementary amenable group, we deduce that the surface group Γ belongs to the class \mathcal{C} , and hence also to the class \mathcal{K} .

Remark 3.5. Let Γ be a cocompact Fuchsian group, namely Γ is a discrete subgroup of $\mathbf{SL}(2, \mathbb{R})$ such that the quotient space $\Gamma \backslash \mathbf{SL}(2, \mathbb{R})$ is compact. Then there is a torsion-free subgroup G of Γ of finite index such that G is the fundamental group of a compact Riemann surface of genus greater than one. By Remark 3.4 above, G is in \mathcal{K} , and since Γ/G is a finite group, it is amenable. Therefore Γ is also in \mathcal{K} .

Remark 3.6. Consider the modular group $\mathbf{SL}(2, \mathbb{Z})$. Then it is well known that there is a congruence subgroup $\Gamma(N)$ of finite index in $\mathbf{SL}(2, \mathbb{Z})$ that is isomorphic to a free group. We conclude by the arguments in Remarks 3.4 and 3.5, that the modular group and all of the congruence subgroups are in \mathcal{K} .

Theorem 3.7. *Every group in \mathcal{K} has the algebraic eigenvalue property.*

The proof of this theorem closely follows the argument in [9] for \mathcal{C} , and we leave the details to section 8.

Remark 3.8. Results of [9] were formulated for operators of the form

$$(13) \quad B = \sum_{g \in S} B(g) L_g$$

acting on $l^2(G)^n$ where $S \subset G$ is a finite subset, $A(g)$ is an $n \times n$ complex matrix and L_g denotes the *untwisted* left translation on $l^2(G)$. Since $x \mapsto x^{-1}$ induces a unitary transformation on $l^2(G)$ that conjugates the right translation R_g with L_g , we see that (13) is unitarily equivalent to

$$(14) \quad \sum_{h \in S} B(h) R_h.$$

It follows that all results of [9] concerning spectral properties of operators (13) apply to operators of the form (14) equally well.

Suppose now we have an operator $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ acting on $l^2(G)^d$ by left twisted multiplication, as described in section 2, where $\overline{\mathbb{Q}}(G, \sigma)$ is the twisted group algebra over the algebraic numbers $\overline{\mathbb{Q}}$ with multiplier σ . For a fixed σ , one can ask if any such A can have transcendental eigenvalues.

Definition 3.9. A discrete group G is said to have the σ -multiplier algebraic eigenvalue property, if for every matrix $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$, regarded as an operator on $l^2(G)^d$, the eigenvalues of A are algebraic numbers, where $\sigma \in Z^2(G, \mathcal{U}(\overline{\mathbb{Q}}))$ is an algebraic multiplier, and $\mathcal{U}(\overline{\mathbb{Q}})$ denotes the unitary elements of the field of algebraic numbers.

An immediate consequence of Lemma 2.7 is that for a given group G , the σ -multiplier algebraic eigenvalue property depends only on the cohomology class of σ .

Corollary 3.10. *Suppose G has the σ -multiplier algebraic eigenvalue property. Then G has the σ' -multiplier algebraic eigenvalue property for any $\sigma' \sim \sigma$ in $Z^2(G, \mathcal{U}(\overline{\mathbb{Q}}))$.*

Proof. Any $A' \in M_d(\overline{\mathbb{Q}}(G, \sigma'))$ is unitarily equivalent to some $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ by Lemma 2.7, and so has only algebraic eigenvalues. \square

In the following sections 4 and 5 we investigate the situation when σ is rational, that is, when $\sigma^n = 1$ for some n . In particular it is shown that every group in \mathcal{K} has the σ -multiplier algebraic eigenvalue property when σ is rational.

The case of more general algebraic multipliers is discussed in section 6.

4. SPECTRAL PROPERTIES WITH RATIONAL σ

Suppose the weight function σ is rational with $\sigma^r = 1$, and let G^σ be the extension of G by \mathbb{Z}_r as follows,

$$(15) \quad \begin{aligned} 1 &\longrightarrow \mathbb{Z}_r \longrightarrow G^\sigma \longrightarrow G \longrightarrow 1 \\ (z_1, g_1) \cdot (z_2, g_2) &= (z_1 z_2 \sigma(g_1, g_2), g_1 g_2) \end{aligned}$$

regarding \mathbb{Z}_r as a (multiplicative) subgroup of $U(1)$. One can then relate the spectrum of an operator $A^\sigma \in M_d(K(G, \sigma))$ acting on $l^2(G)^d$ as in (5) to that of an associated operator \tilde{A} on $l^2(G^\sigma)^d$.

Define a map $\Psi : M_d(K(G, \sigma)) \rightarrow M_d(KG^\sigma)$ as follows. For $A^\sigma \in M_d(K(G, \sigma))$ with matrices of coefficients $A(g) \in M_d(K)$, let $\tilde{A} = \Psi(A^\sigma)$ be given by

$$(16) \quad \tilde{A}(z, g) = \begin{cases} A(g) & \text{if } z = 1, \\ 0 & \text{otherwise,} \end{cases}$$

acting on $l^2(G^\sigma)^d$ by left multiplication.

Consider the map $\xi : l^2(G)^d \rightarrow l^2(G^\sigma)^d$ given by $(\xi f)(z, g) = \bar{z} f(g)$. Then $\frac{1}{\sqrt{r}} \xi$ is an isometry from $l^2(G)^d$ to the closed subspace R of $l^2(G^\sigma)^d$ where $R = \{f \mid f(z, g) = \bar{z} f(1, g) \forall (z, g) \in G^\sigma\}$.

By (15), $(1, g)^{-1} \cdot (z, h) = (z \overline{\sigma(g, g^{-1}h)}, g^{-1}h)$ and so

$$\begin{aligned} (\tilde{A}\xi f)(z, h) &= \sum_{(z', g) \in G^\sigma} \tilde{A}(z', g)(\xi f)((z', g)^{-1} \cdot (z, h)) \\ &= \sum_{g \in G} A(g)(\xi f)(z \overline{\sigma(g, g^{-1}h)}, g^{-1}h) \\ &= \sum_{g \in G} A(g)f(g^{-1}h)\sigma(g, g^{-1}h)\bar{z} \\ &= (\xi A^\sigma f)(z, h), \end{aligned}$$

for all $(z, h) \in G^\sigma$, and thus

$$(17) \quad \Psi(A^\sigma)\xi = \xi A^\sigma \quad \forall A^\sigma \in M_d(K(G, \sigma)).$$

A^σ is therefore unitarily equivalent to the restriction to the subspace R of the operator \tilde{A} .

Lemma 4.1. *Let A be a bounded linear operator on a Hilbert space H , such that $\text{im } A|_R \subset R$ for a closed subspace R of H . Then regarding $A|_R$ as an operator on R , $\text{spec}_{\text{point}} A|_R \subset \text{spec}_{\text{point}} A$ and $\text{spec } A|_R \subset \text{spec } A$.*

Proof. Any eigenvector in R is an eigenvector in H and so the inclusion of point spectrum is immediate.

Suppose $\lambda \notin \text{spec } A$. Then $\lambda \notin \text{spec}_{\text{point}} A|_R$ and $\text{im}(A - \lambda)|_R$ is dense in R . Let B be the inverse of $A - \lambda$. For $u \in R$ one can find a convergent net $u_\alpha \rightarrow u$ with $u_\alpha = (A - \lambda)u'_\alpha$ for u'_α in R . Applying B gives $u'_\alpha \rightarrow Bu$, but R is closed, and so Bu is in R and u is in the image of $(A - \lambda)$. Therefore $(A - \lambda)|_R$ has inverse $B|_R$ and $\lambda \notin \text{spec } A|_R$. \square

We therefore have spectral inclusions for A^σ and \tilde{A} .

Proposition 4.2. *Let $A^\sigma \in M_d(K(G, \sigma))$ be a bounded linear operator on $l^2(G)^d$ as in (5), and suppose σ is rational. Let $\tilde{A} = \Psi(A^\sigma) : l^2(G^\sigma)^d \rightarrow l^2(G^\sigma)^d$ be the associated G^σ -equivariant operator as described above. Then*

$$(18) \quad \text{spec } A^\sigma \subseteq \text{spec } \tilde{A},$$

and

$$(19) \quad \text{spec}_{\text{point}} A^\sigma \subseteq \text{spec}_{\text{point}} \tilde{A}.$$

The following is an easy corollary.

Corollary 4.3. *Let \tilde{A} and A^σ be as described in Proposition 4.2. Then any interval (a, b) that is a gap in the spectrum of \tilde{A} is also contained in a gap of the spectrum of A^σ .*

In section 8 we prove the following result concerning the class of groups \mathcal{K} introduced in section 3.

Proposition 4.4. *The class \mathcal{K} of groups is closed under taking extensions with cyclic kernel.*

Therefore, by Theorem 3.7, the groups in this class all have the σ -multiplier algebraic eigenvalue property for rational σ .

Corollary 4.5 (Absence of eigenvalues that are transcendental numbers). *Any $A^\sigma \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ has only eigenvalues that are algebraic numbers, whenever $G \in \mathcal{K}$ and σ is a rational multiplier on G .*

Proof. Let G^σ be the central extension of the group G in the class \mathcal{K} , where G^σ is defined in (15), and let \tilde{A} be the operator on $l^2(G^\sigma)^d$ associated with A^σ as defined in (16). By Proposition 4.4, any central extension of G by a cyclic group \mathbb{Z}_r is also in the class \mathcal{K} , therefore the group G^σ where σ is in \mathcal{K} . By Theorem 3.7, we know that every group G in the class \mathcal{K} has the algebraic eigenvalue property and so \tilde{A} has only algebraic eigenvalues. A^σ therefore has only algebraic eigenvalues by Proposition 4.2. \square

Recall the definition of the following class of groups from [9].

Definition 4.6. Let \mathcal{G} be the smallest class of groups which contains the trivial group and is closed under the following processes:

- (1) If $H \in \mathcal{G}$ and G is a generalized amenable extension of H , then $G \in \mathcal{G}$.
- (2) If $H \in \mathcal{G}$ and $U < H$, then $U \in \mathcal{G}$.
- (3) If $G = \lim_{i \in I} G_i$ is the direct or inverse limit of a directed system of groups $G_i \in \mathcal{G}$, then $G \in \mathcal{G}$.

We have the inclusion $\mathcal{C} \subset \mathcal{K} \subset \mathcal{G}$, and in particular the class \mathcal{G} contains all amenable groups, free groups, residually finite groups, and residually amenable groups. Consider the subclass of groups $\hat{\mathcal{G}}$ defined as

$$\hat{\mathcal{G}} = \left\{ G \in \mathcal{G} : \hat{G} \in \mathcal{G} \ \forall \ \mathbb{Z}_r\text{-extensions } \hat{G} \text{ of } G \right\}.$$

By the results of section 8, we have the inclusion $\mathcal{C} \subset \mathcal{K} \subset \hat{\mathcal{G}} \subset \mathcal{G}$.

Corollary 4.7 (Absence of eigenvalues that are Liouville transcendental numbers). *Any self-adjoint $A^\sigma \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ does not have any eigenvalues that are Liouville transcendental numbers, whenever $G \in \hat{\mathcal{G}}$ and σ is a rational multiplier on G .*

Proof. Any operator $A^\sigma \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ is self adjoint if and only if $A(g)^* = A(g^{-1})\sigma(g, g^{-1})$ for all $g \in G$. By Lemmas 2.7 and 2.8, there exists an $A' \in M_d(\overline{\mathbb{Q}}(G, \sigma'))$ such that A^σ and A' are unitarily equivalent and where $\sigma'(g, g^{-1}) = 1$ for all $g \in G$. With A^σ being self adjoint, we have that A' is self adjoint and thus that $A'(g)^* = A'(g^{-1})$ for all $g \in G$. By the construction of Lemma 2.8, if σ is a rational multiplier with $\sigma^r = 1$, then σ' is also rational, with $\sigma'^{2r} = 1$.

Let $\tilde{A} = \Psi(A') \in M_d(\overline{\mathbb{Q}}(G^{\sigma'}))$, as in (16), where G^σ is the central extension of G as described in (15). In terms of matrices of coefficients, $\tilde{A}(z, g) = \delta_1(z)A(g)$ for $(z, g) \in G^{\sigma'}$. As $\sigma'(g, g^{-1}) = 1$,

$$\tilde{A}((z, g)^{-1}) = \tilde{A}(z^{-1}\sigma(g, g^{-1}), g^{-1}) = \tilde{A}(z^{-1}, g^{-1})$$

and

$$\tilde{A}(z^{-1}, g^{-1}) = \delta_1(z^{-1})A'(g^{-1}) = \delta_1(z)A'(g)^* = \tilde{A}(z, g)^*,$$

showing that \tilde{A} is self-adjoint.

$\hat{\mathcal{G}}$ is closed under extensions by finite cyclic groups, and so the group $G^{\sigma'}$ is in the class \mathcal{G} . Applying Theorem 4.15 of [9], it follows that \tilde{A} does not have any eigenvalues that are Liouville transcendental numbers. Then by Proposition 4.2, A' and thus A^σ do not have any eigenvalues that are Liouville transcendental numbers. \square

5. ON THE FINITENESS OF THE NUMBER OF DISTINCT EIGENVALUES

We deal here with the following situation: G is a discrete group and $A \in M_d(\overline{\mathbb{Q}}G)$. Then A induces a bounded linear operator $A: l^2(G)^d \rightarrow l^2(G)^d$ by left convolution (using the canonical left G -action on $l^2(G)$), which commutes with the right G -action.

Let $\text{pr}_{\ker A}: l^2(G)^d \rightarrow l^2(G)^d$ denote the orthogonal projection onto $\ker A$. Recall that the von Neumann dimension of $\ker A$ is defined as

$$\dim_G(\ker A) := \text{tr}_G(\text{pr}_{\ker A}) := \sum_{i=1}^d \langle \text{pr}_{\ker A} e_i, e_i \rangle_{l^2(G)^d},$$

where $e_i \in l^2(G)^d$ is the vector with the trivial element of $G \subset l^2(G)$ at the i^{th} -position and zeros elsewhere.

Let G be a discrete group. Let $\text{fin}(G)$ denote the additive subgroup of \mathbb{Q} generated by the inverses of the orders of the finite subgroups of G . Note that $\text{fin}(G) = \mathbb{Z}$ if and only if G is torsion free and $\text{fin}(G)$ is discrete in \mathbb{R} if and only if orders of finite subgroups of G are bounded. Recall the following definition.

Definition 5.1. A discrete group G is said to fulfill the *strong Atiyah conjecture* if the orders of the finite subgroups of G are bounded and

$$\dim_G(\ker A) \in \text{fin}(G) \quad \forall A \in M_d(\overline{\mathbb{Q}}G);$$

where $\ker A$ is the kernel of the induced map $A: l^2(G)^d \rightarrow l^2(G)^d$.

Linnell proved the strong Atiyah conjecture if G is such that the orders of the finite subgroups of G are bounded and $G \in \mathcal{C}$, where \mathcal{C} is Linnell's class of groups that is defined just below Definition 3.1.

Theorem 5.2 ([14]). *If $G \in \mathcal{C}$ is such that the orders of the finite subgroups of G are bounded, then the strong Atiyah conjecture is true.*

In [9], Linnell's results were generalized to a larger class \mathcal{D} of groups, but these groups are all torsion-free and therefore our results do not apply to them.

The following is an easy corollary of the relationship between A^σ and \tilde{A} , cf. (15).

Corollary 5.3 (Finite number of distinct eigenvalues). *Any self-adjoint $A^\sigma \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ has only a finite number of distinct eigenvalues whenever $G \in \mathcal{C}$ is such that the orders of the finite subgroups of G are bounded, and σ is a rational multiplier on G .*

Proof. Let G^σ be the central extension of the group G in \mathcal{C} , where G^σ is defined in (15), and let \tilde{A} be the operator on $l^2(G^\sigma)$ associated with A^σ as defined in (16). By Remark 8.5, the group G^σ as defined in (15) is also in \mathcal{C} . Clearly, the orders of finite subgroups of G^σ are bounded as well. Thus the Theorem 5.2 above applies to G^σ , and so the dimensions of eigenspaces of \tilde{A} are in the discrete, closed subgroup $\text{fin}(G)$ of \mathbb{R} and are therefore bounded away from zero. It follows that \tilde{A} can have at most finitely many eigenvalues. The conclusion now follows from (19). \square

Remark 5.4. Our results do not just apply to operators acting on scalar valued functions but also to vector valued functions. In this case there are many examples where eigenvalues exist. For instance, for the combinatorial Laplacian on L^2 , degree zero cochains of a covering space, zero is never an eigenvalue, whereas it is very common for it to be an eigenvalue on L^2 cochains of positive

degree. This is the case whenever the Euler characteristic of the base is nonzero, which follows for instance from Atiyah's L^2 index theorem for covering spaces [2], and Dodziuk's theorem on the combinatorial invariance of the L^2 Betti numbers, [8].

Let H denote the lamplighter group, namely H is the wreath product of \mathbb{Z}_2 and \mathbb{Z} . Then there is the following remarkable computation of Grigorchuk and Żuk, Theorem 2 and Corollary 3, [13].

Theorem 5.5. *Let $A := t + at + t^{-1} + (at)^{-1} \in \mathbb{Z}H$ be a multiple of the Random Walk operator of H . Then A , considered as an operator on $l^2(H)$, has eigenvalues*

$$(20) \quad \left\{ 4 \cos \left(\frac{p}{q} \pi \right) \mid p \in \mathbb{Z}, q = 2, 3, \dots \right\}.$$

The L^2 -dimension of the corresponding eigenspaces is

$$(21) \quad \dim_H \ker \left(A - 4 \cos \left(\frac{p}{q} \pi \right) \right) = \frac{1}{2^q - 1} \quad \text{if } p, q \in \mathbb{Z}, q \geq 2, \text{ with } (p, q) = 1.$$

Note that the number of distinct eigenvalues of A is infinite and dense in some interval! However, the orders of the torsion subgroups of H is unbounded so that Corollary 5.3 is not contradicted. The eigenvalues of A are algebraic numbers as predicted by our Theorem 2.5 in [18]. This can be seen as follows: since $(\cos(p/q\pi) + i \sin(p/q\pi))^q = (-1)^p$, this shows that $\cos(p/q\pi) + i \sin(p/q\pi)$ is an algebraic number. Therefore the real part, $\cos(p/q\pi)$, is also an algebraic number.

6. THE CASE OF ALGEBRAIC MULTIPLIERS

The goal in this section is to extend the results that were obtained in the previous sections, from rational multipliers to the more general case of algebraic multipliers. Recall that a generic algebraic multiplier is not necessarily a rational multiplier. We start with an example of algebraic numbers on the unit circle that are not roots of unity. Consider the roots of the polynomial,

$$(22) \quad z^4 - z^3 + (2 - k)z^2 - z + 1 = \left(z^2 - \frac{(1 + \sqrt{4k + 1})}{2} \cdot z + 1 \right) \cdot \left(z^2 - \frac{(1 - \sqrt{4k + 1})}{2} \cdot z + 1 \right)$$

with k a positive integer. This polynomial is irreducible over \mathbb{Z} if $4k + 1$ is not a square. We look for k such that the first factor has two distinct real roots while the second one has two complex conjugate roots. Thus we seek k so that

$$\frac{(1 + \sqrt{4k + 1})^2}{4} - 4 > 0 \quad \text{and} \quad \frac{(1 - \sqrt{4k + 1})^2}{4} - 4 < 0.$$

It is easily seen that the only values of k satisfying these conditions are $k = 3, 4, 5$. For each of these choices, two of the roots, denoted by $e^{i\theta}$ and $e^{-i\theta}$, lie on the unit circle and are roots of the second factor in (22). The two other roots are real, denoted by r and r^{-1} , where $r < 1$. The numbers $e^{i\theta}$, $e^{-i\theta}$, r and r^{-1} are algebraic integers, which are all conjugate to each other. Therefore $e^{i\theta}$ is not a root of unity since otherwise all its conjugates would also be roots of unity. However, the numbers $e^{i\theta}$, $e^{-i\theta}$, r and r^{-1} are units in the corresponding ring of algebraic integers. Since $e^{i\theta}$ is not a root of unity, its powers are dense in the unit circle whereas the positive powers of r tend to 0. For fixed $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\theta = \alpha_2 - \alpha_1$, and for all $(m, n) \in \mathbb{Z}^2$, let

$$(23) \quad \sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 n' m)).$$

Then σ is an algebraic multiplier on \mathbb{Z}^2 whose cohomology class $[\sigma] \in H^2(\mathbb{Z}^2, U(1)) \cong U(1)$ is equal to $e^{i\theta}$, so that σ is *not* a rational multiplier. It is well known that σ determines the noncommutative torus A_θ , see [4].

The trivial group has the σ -multiplier algebraic eigenvalue property for any σ , since the eigenvalues are the zeros of the characteristic polynomial. The same is true for every finite group. If G has the σ -multiplier algebraic eigenvalue property and H is a subgroup of G , then H also has the σ -multiplier algebraic eigenvalue property.

Theorem 6.1. *Every free group has the σ -multiplier algebraic eigenvalue property for every σ .*

Proof. Let G be a free group. Then G has the algebraic eigenvalue property, corresponding to the identity multiplier, by Theorem 4.5 of [9]. However for a free group, every multiplier is cohomologous to the identity, as free groups have no cohomology of degree two or higher. This can be seen by noting that the classifying space of a free group is a bouquet of circles, and so is one dimensional (see for example [5, Chapter II, Section 4, Example 1].) By Corollary 3.10 then, G has the σ -multiplier algebraic eigenvalue for every algebraic multiplier σ . \square

Theorem 6.2. *Suppose that we have a short exact sequence of groups*

$$(24) \quad 1 \rightarrow H \rightarrow G \xrightarrow{p} G/H \rightarrow 1$$

where the quotient group G/H is a finitely generated amenable group. Let σ' be an algebraic multiplier on G/H , and let $\sigma = p^\sigma'$ be the pullback of σ' . Then if H has the algebraic eigenvalue property, G has the σ -multiplier algebraic eigenvalue property.*

Proof. We will show that an operator $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ has only algebraic eigenvalues by first demonstrating that the point spectrum of A is a subset of the union of the point spectra of a series $A^{(m)}$ of approximations to A , and then showing that each $A^{(m)}$ is equivalent to the untwisted action of a matrix over $\overline{\mathbb{Q}}H$, and thus has only algebraic eigenvalues.

In the following let \mathcal{A} be the von Neumann algebra $W_L^*(G, \sigma) \otimes M_d(\mathbb{C})$, with trace $\tau = \text{tr}_{G, \sigma}$ as defined in Corollary 2.4. For finite $X \subset G/H$ let \mathcal{H}_X be the subspace $l^2(p^{-1}(X))^d$ of $l^2(G)^d$ and let \mathcal{A}_X be the commutant $B(\mathcal{H}_X)^H$ of the right H -translations on \mathcal{H}_X . Picking a right inverse s of p , the isometry

$$\begin{aligned} \iota_X : \mathcal{H}_X &= l^2(p^{-1}(X))^d \rightarrow l^2(H)^{d, \#X} \\ (\iota_X f)(h)_{a,i} &= f(s(x_i)h)_a \quad \text{for } h \in H, x_i \in X, a = 1, \dots, d \end{aligned}$$

induces an isomorphism ψ_X from \mathcal{A}_X to $W_L^*(H) \otimes M_d(\mathbb{C}) \otimes M_{\#X}(\mathbb{C})$. Define a trace τ_X on \mathcal{A}_X by

$$\tau_X(B) = \frac{1}{\#X} (\text{tr}_H \otimes \text{Tr})(\psi_X B)$$

where tr_H is the usual trace on $W_L^*(H)$ and Tr is the canonical matrix trace on $M_d(\mathbb{C}) \otimes M_{\#X}(\mathbb{C})$. In terms of the components $(B_{a,b})_{g,k}$ of an operator $B \in \mathcal{A}_X$ (for $g, k \in G$ and $a, b = 1, \dots, d$), the trace is given by

$$\tau_X(B) = \frac{1}{\#X} \sum_{a=1}^d \sum_{x \in X} (B_{a,a})_{s(x), s(x)}.$$

The von Neumann dimension associated with τ_X will be denoted by \dim_X .

The multiplier $\sigma(g, h) = 1$ for all $h \in H$, so any operator $A \in \mathcal{A}$ commutes with the right H -translations. For $A \in \mathcal{A}$ and $X \subset G/H$ let $A^{(X)} = P_X A|_{\mathcal{H}_X}$ where P_X is the orthogonal projection onto \mathcal{H}_X . $A^{(X)}$ then belongs to \mathcal{A}_X and $\tau_X(A^{(X)}) = \tau(A)$.

The dimension functions on the algebras \mathcal{A}_X satisfy the following easily verifiable relations, for finite subsets $X \subseteq Y$:

$$(25) \quad \dim_X L \leq \dim_\tau N \quad \text{for all } L\eta\mathcal{A}_X, N\eta\mathcal{A} \text{ with } L \subset N,$$

$$(26) \quad \dim_X L = \frac{\#Y}{\#X} \dim_Y L \quad \text{for all } L\eta\mathcal{A}_X,$$

$$(27) \quad \dim_Y M \geq \dim_Y P_X(M) \quad \text{for all } M\eta\mathcal{A}_Y.$$

As G/H is amenable and finitely generated, it admits a Følner exhaustion by finite subsets $\{X_m\}$ such that

$$\lim_{m \rightarrow \infty} \frac{\#\partial X_m}{\#X_m} = 0,$$

where ∂X_m is the δ -neighbourhood (with regard to the word metric on G/H) of X_m for any fixed δ .

In the following, let $A \in M_d(\overline{\mathbb{Q}}(G, \sigma))$ and let $A^{(m)} = A^{(X_m)} \in \mathcal{A}_{X_m}$. Suppose λ is not an eigenvalue of any of the $A^{(m)}$, and consider the space E_λ of λ -eigenfunctions of A with corresponding orthogonal projection P_λ . For any finite $X \subset G/H$,

$$\begin{aligned} \dim_\tau E_\lambda &= \tau(P_\lambda) = \tau_X(P_X P_\lambda|_{\mathcal{H}_X}) \\ &\leq \dim_X \operatorname{im} P_X P_\lambda|_{\mathcal{H}_X} \quad (\text{as } \|P_X P_\lambda\| \leq 1) \\ &\leq \dim_X P_X(E_\lambda). \end{aligned}$$

As A is a matrix over the twisted group algebra, each component is a finite sum of twisted translations, and consequently A has *bounded propagation*. Explicitly, there are only a finite number of the matrices of coefficients $A(g) \in M_d(\mathbb{C})$ which are non-zero, and so we can choose a bound κ by

$$\kappa = \max\{d_{G/H}(1_{G/H}, gH) \mid A(g) \neq 0\}$$

(where $d_{G/H}$ is the word-metric on G/H) so that for f with support in $p^{-1}(X)$, Af will have support in $p^{-1}(X')$, where $X' = \{x \mid d_{G/H}(x, X) \leq \kappa\}$ is the κ -neighbourhood of X_m .

Let X'_m be the κ -neighbourhood of X_m , and $\partial X_m = X'_m \setminus X_m$ be the outer κ -boundary of X_m . Then

$$P_{X_m} A = P_{X_m} A P_{X'_m} = A^{(m)} P_{X_m} + P_{X_m} A P_{\partial X_m}.$$

For $f \in E_\lambda$ with $P_{\partial X_m} f = 0$ then, $P_{X_m} A f = \lambda P_{X_m} f = A^{(m)} P_{X_m} f$. By assumption though, λ is not an eigenvalue of $A^{(m)}$, and so

$$(28) \quad f \in E_\lambda \text{ and } P_{\partial X_m} f = 0 \implies P_{X_m} f = 0.$$

Picking some superset Y of X'_m , one has $P_{\partial X_m} P_Y = P_{\partial X_m}$ and $P_{X_m} P_Y = P_{X_m}$, and so (28) implies

$$(29) \quad \ker P_{\partial X_m}|_{P_Y(E_\lambda)} \subseteq \ker P_{X_m}|_{P_Y(E_\lambda)}.$$

Applying Lemma 2.6 gives

$$\begin{aligned}\dim_Y P_Y(E_\lambda) &= \dim_Y \ker P_{\partial X_m}|_{P_Y(E_\lambda)} + \dim_Y P_{\partial X_m}(E_\lambda) \\ &= \dim_Y \ker P_{X_m}|_{P_Y(E_\lambda)} + \dim_Y P_{X_m}(E_\lambda),\end{aligned}$$

which with the inclusion (29) in turn gives

$$\dim_Y P_{X_m}(E_\lambda) \leq \dim_Y P_{\partial X_m}(E_\lambda) \leq \dim_Y \operatorname{im} P_{\partial X_m} = \frac{\#\partial X_m}{\#Y}.$$

Then for any m and $Y \supseteq X'_m$,

$$\dim_\tau E_\lambda \leq \dim_{X_m} P_{X_m}(E_\lambda) = \frac{\#Y}{\#X_m} \dim_Y P_{X_m}(E_\lambda) \leq \frac{\#\partial X_m}{\#X_m},$$

which goes to zero as m goes to infinity, as the X_m form a Følner exhaustion of G/H . Consequently $\dim_\tau E_\lambda = 0$ and λ is not an eigenvalue of A ; that is, any eigenvalue of A must be an eigenvalue of $A^{(m)}$ for some m .

For $f \in \mathcal{H}_X$, let $\hat{f} = \iota_X f$ be the corresponding element in $l^2(H)^{d, \#X}$ defined by $\hat{f}(h)_{a,i} = f(s(x_i)h)_a$, as before. Then for every $x_i \in X$

$$\begin{aligned}(Af)(s(x_i)h) &= \sum_{j=1}^{\#X} \sum_{k \in H} A(s(x_i)hk^{-1}s(x_j)^{-1})f(s(x_i)k)\sigma(s(x_i)hk^{-1}s(x_j)^{-1}, s(x_j)k) \\ &= \sum_{j=1}^{\#X} \sum_{k \in H} A(s(x_i)hk^{-1}s(x_j)^{-1})f(s(x_i)k)\sigma'(x_i x_j^{-1}, x_j) \\ &= (B\hat{f})(h)_i \in \mathbb{C}^d,\end{aligned}$$

where the matrix of coefficients of h in $B \in M_{(d\#X)}(\overline{\mathbb{Q}}H)$ is given by

$$B(h)_{(a,i),(b,j)} = A(s(x_i)hs(x_j)^{-1})_{a,b} \cdot \sigma'(x_i x_j^{-1}, x_j)$$

for $i, j = 1, \dots, \#X$ and $a, b = 1, \dots, d$. As H has the algebraic eigenvalue property, B and hence A^X have only algebraic eigenvalues. Consequently, the operators $A^{(m)}$ and A have only algebraic eigenvalues. \square

One of the main theorems in this section is the following.

Theorem 6.3. *Let Γ be the fundamental group of a closed Riemann surface of genus $g > 1$. Then Γ has the σ -multiplier algebraic eigenvalue property, where σ is any algebraic multiplier on Γ .*

We want to use Theorem 6.2 using the exact sequence of Remark (3.4). To do this, it is necessary to prove that every algebraic multiplier σ on Γ is cohomologous to the pull-back of an algebraic multiplier σ' on \mathbb{Z}^{2g} . The construction of σ' was used in [6, Section 7.2]. We follow it closely paying particular attention to algebraicity.

Recall that the *area cocycle* c of the fundamental group of a compact Riemann surface, $\Gamma = \Gamma_g$ is a canonically defined 2-cocycle on Γ that is defined as follows. Firstly, recall the definition of a well known area 2-cocycle on $\mathbf{PSL}(2, \mathbb{R})$. $\mathbf{PSL}(2, \mathbb{R})$ acts on \mathbb{H} so that $\mathbb{H} \cong \mathbf{PSL}(2, \mathbb{R})/\mathbf{SO}(2)$. Then

$$c(\gamma_1, \gamma_2) = \operatorname{Area}_{\mathbb{H}}(\Delta(o, \gamma_1 \cdot o, \gamma_1 \gamma_2 \cdot o)),$$

where o denotes an origin in \mathbb{H} and $\text{Area}_{\mathbb{H}}(\Delta(a, b, c))$ is the oriented hyperbolic area of the geodesic triangle in \mathbb{H} with vertices at $a, b, c \in \mathbb{H}$. The restriction of c to the subgroup Γ is the area cocycle c of Γ . We use the additive notation when discussing area cocycles and remark that $(2\pi)^{-1}c$ represents an integral class in $H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}$ as follows from Gauss-Bonnet theorem.

Let Ω_j denote the (diagonal) operator on $l^2(\Gamma)$ defined by

$$\Omega_j f(\gamma) = \Omega_j(\gamma) f(\gamma) \quad \forall f \in l^2(\Gamma) \quad \forall \gamma \in \Gamma$$

where

$$\Omega_j(\gamma) = \int_o^{\gamma \cdot o} \alpha_j \quad j = 1, \dots, 2g$$

and where

$$(30) \quad \{\alpha_j\}_{j=1}^{2g} = \{a_j\}_{j=1}^g \cup \{b_j\}_{j=1}^g$$

is a collection of harmonic 1-forms on the compact Riemann surface $\Sigma_g = \mathbb{H}/\Gamma$, generating $H^1(\Sigma_g, \mathbb{R}) = \mathbb{R}^{2g}$. We abuse the notation slightly and do not distinguish between a form on Σ_g and its pullback to the hyperbolic plane as well as between an element of Γ and a loop in Σ_g representing it.

Notice that we can write equivalently

$$\Omega_j(\gamma) = c_j(\gamma),$$

where the group cocycles c_j form a symplectic basis for $H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^{2g}$, with generators $\{\alpha_j\}_{j=1, \dots, 2g}$, as in (30) and can be defined as the integration on loops on Σ_g ,

$$c_j(\gamma) = \int_{\gamma} \alpha_j.$$

Define

$$\Psi_j(\gamma_1, \gamma_2) = \Omega_j(\gamma_1) \Omega_{j+g}(\gamma_2) - \Omega_{j+g}(\gamma_1) \Omega_j(\gamma_2).$$

Let $\Xi : \mathbb{H} \rightarrow \mathbb{R}^{2g}$ denote the Abel-Jacobi map

$$\Xi : x \mapsto \left(\int_o^x a_1, \int_o^x b_1, \dots, \int_o^x a_g, \int_o^x b_g \right),$$

where \int_o^x means integration along the unique geodesic in \mathbb{H} connecting o to x . Having chosen an origin o once and for all we make an identification $\Gamma \cdot o \cong \Gamma$. Note that Γ acts on \mathbb{R}^{2g} in a natural way and the map Ξ is Γ -equivariant. In addition, the map Ξ is a symplectic map, that is, if ω and ω_J are the respective symplectic 2-forms, then one has $\Xi^*(\omega_J) = k\omega$ for a suitable constant k . Henceforth, we renormalize ω (and consequently the area cocycle c) so that $\Xi^*(\omega_J) = \omega$. One then has the following geometric lemma [6], [17].

Lemma 6.4.

$$\Psi(\gamma_1, \gamma_2) = \sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2) = \int_{\Delta_E(\gamma_1, \gamma_2)} \omega_J$$

where $\Delta_E(\gamma_1, \gamma_2)$ denotes the Euclidean triangle with vertices at $\Xi(o), \Xi(\gamma_1 \cdot o)$ and $\Xi(\gamma_1 \gamma_2 \cdot o)$, and ω_J denotes the flat Kähler 2-form on the universal cover of the Jacobi variety. That is, $\sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2)$ is equal to the Euclidean area of the Euclidean triangle $\Delta_E(\gamma_1, \gamma_2)$.

That is, the cocycle $\Psi = p^*(\Psi')$, where Ψ' is a 2-cocycle on \mathbb{Z}^{2g} and p is defined as the projection.

$$1 \rightarrow F \rightarrow \Gamma \xrightarrow{p} \mathbb{Z}^{2g} \rightarrow 1$$

The following lemma is also implicit in [6], [17].

Lemma 6.5. *The hyperbolic area group 2-cocycle c and the Euclidean area group 2-cocycle Ψ on Γ , are cohomologous.*

Proof. Observe that since $\omega = \Xi^*\omega_J$, one has

$$c(\gamma_1, \gamma_2) = \int_{\Delta(\gamma_1, \gamma_2)} \omega = \int_{\Xi(\Delta(\gamma_1, \gamma_2))} \omega_J.$$

Therefore the difference

$$\begin{aligned} \Psi(\gamma_1, \gamma_2) - c(\gamma_1, \gamma_2) &= \int_{\Delta_E(\gamma_1, \gamma_2)} \omega_J - \int_{\Xi(\Delta(\gamma_1, \gamma_2))} \omega_J \\ &= \int_{\partial\Delta_E(\gamma_1, \gamma_2)} \Theta_J - \int_{\partial\Xi(\Delta(\gamma_1, \gamma_2))} \Theta_J, \end{aligned}$$

where Θ_J is a 1-form on the universal cover \mathbb{R}^{2g} of the Jacobi variety such that $d\Theta_J = \omega_J$.

Let $h(\gamma) = \int_{\Xi(\ell(\gamma))} \Theta_J - \int_{m(\gamma)} \Theta_J$, where $\ell(\gamma)$ denotes the unique geodesic in \mathbb{H} joining o and $\gamma \cdot o$ and $m(\gamma)$ is the straight line in the Jacobi variety joining the points $\Xi(o)$ and $\Xi(\gamma \cdot o)$. We can also write $h(\gamma) = \int_{D(\gamma)} \omega_J$, where $D(\gamma)$ is an arbitrary topological disk in \mathbb{R}^{2g} with boundary $\Xi(\ell(\gamma)) \cup m(\gamma)$. Thus the equality above can be rewritten as

$$\begin{aligned} \Psi(\gamma_1, \gamma_2) - c(\gamma_1, \gamma_2) &= h(\gamma_1) - h(\gamma_1\gamma_2) + \int_{\gamma_1 \cdot \Xi(\ell(\gamma_2))} \Theta_J - \int_{\gamma_1 \cdot m(\gamma_2)} \Theta_J \\ &= h(\gamma_1) - h(\gamma_1\gamma_2) + \int_{D(\gamma_2)} (\gamma_1)^* d\Theta_J \\ &= h(\gamma_1) - h(\gamma_1\gamma_2) + h(\gamma_2) \\ &= \delta h(\gamma_1, \gamma_2) \end{aligned}$$

since $\omega_J = d\Theta_J$ is invariant under the action of Γ .

□

Lemma 6.6. *Let Γ be the fundamental group of a closed, genus g Riemann surface and*

$$1 \rightarrow F \rightarrow \Gamma \xrightarrow{p} \mathbb{Z}^{2g} \rightarrow 1$$

where F is a free group and \mathbb{Z}^{2g} the free abelian group as in Remark 3.4. Then every multiplier σ on Γ is cohomologous to a multiplier $\sigma' = p^(\sigma'')$ on Γ where σ'' is a multiplier on \mathbb{Z}^{2g} .*

In addition, every algebraic multiplier σ on Γ is cohomologous to an algebraic multiplier $\sigma' = p^(\sigma'')$ on Γ where σ'' is an algebraic multiplier on \mathbb{Z}^{2g} . More precisely, if $\sigma \in Z^2(\Gamma, \mathcal{U}(\overline{\mathbb{Q}}))$ then σ'' can be chosen from $Z^2(\mathbb{Z}^{2g}, \mathcal{U}(\overline{\mathbb{Q}}))$ so that σ and σ' are cohomologous in $Z^2(\Gamma, \mathcal{U}(\overline{\mathbb{Q}}))$.*

Proof. Observe that $\Xi(\Gamma \cdot o) \subset \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$. It follows that the Euclidean area cocycle and its pullback represent integral cohomology classes. By the lemma above, the cohomology class of c is integral. Now let σ be an arbitrary multiplier on Γ . Since $H^2(\Gamma, A) = A$ for every abelian group A and $H^3(\Gamma, \mathbb{Z}) = 0$, we see that σ is cohomologous to a multiplier $\sigma_1 = \exp(2\pi i \theta c)$, where θ is a

real number. By Lemma 6.5 we see that σ_1 is cohomologous to $p^*(\sigma'')$, where $\sigma'' = \exp(2\pi i\theta\Psi')$ is a multiplier on \mathbb{Z}^{2g} .

To prove the last claim, we identify the group cohomology with the cohomology of the surface $\Sigma_g = \mathbb{H}/\Gamma$. Since the value of the cocycle on the fundamental class depends only on the cohomology class and $c(\Sigma_g) = 2g - 2$, we see that $\sigma'(\Sigma_g) = \exp(2\pi i\theta)^{2g-2} = \sigma(\Sigma_g)$ is algebraic. It follows that $\exp(2\pi i\theta)$ is an algebraic number so that σ'' is an algebraic cocycle. Now both σ and σ' are algebraic cocycles. They are cohomologous in $Z^2(\Gamma, U(1))$. For any coefficients the cohomology class of the cocycle is determined by the value of the cocycle on the fundamental class. Therefore σ and σ' represent the same cohomology class in $H^2(\Gamma, \mathcal{U}(\overline{\mathbb{Q}}))$ i.e. are cohomologous in $Z^2(\Gamma, \mathcal{U}(\overline{\mathbb{Q}}))$. \square

Proof of Theorem 6.3. Recall that if two multipliers σ, σ' on Γ are cohomologous, then Γ has the σ -multiplier algebraic eigenvalue property if and only if Γ has the σ' -multiplier algebraic eigenvalue property (Corollary 3.10.) Since the free group F has the algebraic eigenvalue property, and since \mathbb{Z}^{2g} is a finitely generated amenable group, by applying Theorem 6.2 and Lemma 6.6, we deduce Theorem 6.3. \square

7. GENERALIZED INTEGRATED DENSITY OF STATES AND SPECTRAL GAPS

In this section, we will realize the von Neumann trace on the group von Neumann algebra of a surface group, as a generalized integrated density of states, which is an important step to relating it directly to the physics of the quantum Hall effect.

Our first main theorem is the following.

Theorem 7.1. *Consider the situation of Theorem 6.2, where we have a short exact sequence of groups*

$$(31) \quad 1 \rightarrow H \rightarrow G \xrightarrow{p} G/H \rightarrow 1$$

where the quotient group G/H is finitely generated and amenable. Let σ' be a multiplier on G/H , and let $\sigma = p^*\sigma'$ be the pullback of σ' . Let $A \in M_d(\mathbb{C}(G, \sigma))$ be a self-adjoint operator acting on $l^2(G)^d$, being a member of the von Neumann algebra $\mathcal{A} = W_L^*(G, \sigma) \otimes M_d(\mathbb{C})$ with trace $\tau = \text{tr}_{G, \sigma}$.

For finite subsets X of G/H , let $\mathcal{H}_X = l^2(p^{-1}(X))^d$ be the space of functions with support on $p^{-1}(X)$, and let $\mathcal{A}_X = B(\mathcal{H}_X)^H$ be the commutant of the right H -translations on \mathcal{H}_X . Pick a right inverse s of the projection p and give \mathcal{A}_X the trace τ_X as in the proof of Theorem 6.2, which in terms of the components $(B_{a,b})_{g,k}$ of an operator $B \in \mathcal{A}_X$ (for $g, k \in G$ and $a, b = 1, \dots, d$) is given by

$$\tau_X(B) = \frac{1}{\#X} \sum_{a=1}^d \sum_{x \in X} (B_{a,a})_{s(x), s(x)}.$$

Let $A^{(X)} = P_X A|_{\mathcal{H}_X} \in \mathcal{A}_X$ where P_X is the orthogonal projection onto \mathcal{H}_X .

Choose a Følner exhaustion X_m of G/H . Then the spectral density function of A equals the generalised integrated density of states as given by the (normalised) spectral density functions of the operators $A^{(m)} = A^{(X_m)}$. That is, with spectral density functions F of A and F_m of the A_m ,

$$F_m(\lambda) = \tau_{X_m}(\chi_{(-\infty, \lambda]}(A^{(m)})), \quad F(\lambda) = \tau(\chi_{(-\infty, \lambda]}(A)),$$

the F_m converge point-wise to F at every λ ,

$$(32) \quad \lim_{m \rightarrow \infty} F_m(\lambda) = F(\lambda) \quad \forall \lambda \in \mathbb{R}.$$

The proof of this theorem in the case of $H = 1$ was given in [18] and [19] for the discrete magnetic Laplacian. To establish this theorem in our more general situation, we apply the same arguments, slightly generalized as follows, relying upon the notation established in the proof of Theorem 6.2.

Lemma 7.2. *For any polynomial p*

$$\lim_{m \rightarrow \infty} \tau_m(p(A^{(m)})) = \tau(p(A)).$$

Proof. The argument is exactly that of [18], Lemma 2.1, and relies upon the amenability of G/H . \square

Lemma 7.3. *Suppose $f(\lambda)$ and $f_m(\lambda)$ ($m = 1, 2, \dots$) are monotonically increasing right continuous functions on \mathbb{R} that are zero for $\lambda < a$ and constant for $\lambda \geq b$, for fixed a and b . Further suppose that*

$$(33) \quad \lim_{m \rightarrow \infty} \int p \, df_m = \int p \, df$$

for all polynomials p , where the integrals are Lebesgue-Stieltjes integrals. Then

$$f(\lambda) = \underline{f}^+(\lambda) = \overline{f}^+(\lambda)$$

for all λ , where \overline{f}^+ and \underline{f}^+ are defined in terms of the f_m by

$$(34) \quad \begin{aligned} \underline{f}(\lambda) &= \liminf_m f_m(\lambda), & \underline{f}^+(\lambda) &= \lim_{\epsilon \rightarrow 0^+} \underline{f}(\lambda + \epsilon), \\ \overline{f}(\lambda) &= \limsup_m f_m(\lambda), & \overline{f}^+(\lambda) &= \lim_{\epsilon \rightarrow 0^+} \overline{f}(\lambda + \epsilon). \end{aligned}$$

In particular $f(\lambda) = \lim_{m \rightarrow \infty} f_m(\lambda)$ at all points of continuity of f , which is at all but a countable number of points.

Proof. The proof follows that of part (i) of Theorem 2.6 of [18].

Take a sequence of successively closer polynomial approximations p_j to the characteristic function $\chi_{(-\infty, x]}$ over the interval $[a, b]$ such that

$$\chi_{(-\infty, x]}(\lambda) \leq p_j(\lambda) \leq \chi_{(-\infty, x + \frac{1}{j}]}(\lambda) + \frac{1}{j} \quad \forall \lambda \in [a, b], \quad j \geq 1.$$

Then for all j ,

$$(35) \quad f_m(x) \leq \int_a^b p_j(\lambda) df_m(\lambda) \leq f_m(x + \frac{1}{j}) + \frac{1}{j}(b - a),$$

$$(36) \quad f(x) \leq \int_a^b p_j(\lambda) df(\lambda) \leq f(x + \frac{1}{j}) + \frac{1}{j}(b - a).$$

Taking the limit as m goes to infinity, equations (35) and (33),

$$(37) \quad \overline{f}(x) \leq \int_a^b p_j(\lambda) df(\lambda) \leq \underline{f}(x + \frac{1}{j}) + \frac{1}{j}(b - a) \quad \forall j \geq 1.$$

The right continuity of f with equation (36) gives

$$\lim_{j \rightarrow \infty} \int_a^b p_j(\lambda) df(\lambda) = f(x),$$

and so taking the limit of (37) as j goes to infinity gives

$$(38) \quad \overline{f}(x) \leq f(x) \leq \underline{f}^+(x) \quad \forall x.$$

Again using the right continuity of f ,

$$f(x) \leq \underline{f}^+(x) \leq \overline{f}^+(x) \leq f^+(x) = f(x).$$

$f(x)$ is monotonically increasing in x and bounded, so can have at most a countable number of discontinuities. If f is continuous at x then equation (38) implies that $\underline{f}(x) = f(x) = \overline{f}(x)$. \square

Lemma 7.4. *Let F and F_m be as in the statement of Theorem 7.1. Then using the notation (34) of Lemma 7.3,*

$$F(\lambda) = \overline{F}^+(\lambda) = \underline{F}^+(\lambda) \quad \forall \lambda \in \mathbb{R},$$

with

$$(39) \quad \lim_{m \rightarrow \infty} F_m(\lambda) = F(\lambda) \quad \forall \lambda \in \mathbb{R} \text{ such that } F \text{ is continuous at } \lambda.$$

Proof. This is an immediate consequence of the two preceding lemmas. \square

The convergence (39) can be extended to all λ by showing that the jumps of the spectral density functions at points of discontinuity also converge.

Lemma 7.5 (Corollary 3.2 of [19]). *Let f and f_m (for $m = 1, 2, \dots$) be monotonically increasing right continuous functions on \mathbb{R} satisfying $f(\lambda) = \underline{f}^+(\lambda) = \overline{f}^+(\lambda)$ at all λ , as in Lemma 7.3. Denote the jumps at λ of f and the f_m by j and j_m respectively,*

$$j_m(\lambda) = \lim_{\epsilon \rightarrow 0^+} f_m(\lambda) - f_m(\lambda - \epsilon),$$

$$j(\lambda) = \lim_{\epsilon \rightarrow 0^+} f(\lambda) - f(\lambda - \epsilon).$$

Suppose the j_m converge to j point-wise at all λ . Then the f_m converge to f point-wise at all λ .

To obtain point-wise convergence of f_m to f at every point, it is in fact sufficient to show that $\liminf_m j_m(\lambda) \geq j(\lambda)$ at all λ , due to the following lemma.

Lemma 7.6. *Let f and f_m (for $m = 1, 2, \dots$) be monotonically increasing right continuous functions on \mathbb{R} satisfying $f(\lambda) = \underline{f}^+(\lambda) = \overline{f}^+(\lambda)$ at all λ , as in Lemma 7.3. Denote the jumps at λ of f and f_m by j and j_m respectively, as in Lemma 7.5. Then*

$$\limsup_m j_m(\lambda) \leq j(\lambda) \quad \forall \lambda \in \mathbb{R}.$$

Proof. Fix λ . By monotonicity,

$$(40) \quad j_m(\lambda) \leq f_m(\lambda + \epsilon) - f_m(\lambda - \epsilon) \quad \forall \epsilon > 0.$$

f is continuous at all but a countable number of points, and at points x of continuity, $f_m(x) \rightarrow f(x)$ as $m \rightarrow \infty$. Pick a decreasing sequence $\epsilon_k \rightarrow 0$ such that f is continuous at $\lambda + \epsilon_k$ and $\lambda - \epsilon_k$ for all k . Then taking the limit in m of (40) gives

$$\limsup_m j_m(\lambda) \leq f(\lambda + \epsilon_k) - f(\lambda - \epsilon_k) \quad \forall k.$$

By right continuity of f , $f(\lambda + \epsilon_k) - f(\lambda - \epsilon_k)$ converges to $j(\lambda)$ from above as k goes to infinity. Thence on taking the limit in k , $\limsup_m j_m(\lambda) \leq j(\lambda)$. \square

Now consider the situation of Theorem 7.1. We already have a weak spectral approximation by virtue of Lemma 7.4, so all we require now is to show convergence of the jumps in F_m to those of F .

Theorem 7.7. *Let $D(\lambda)$ and $D_m(\lambda)$ denote the jumps at λ of the spectral density functions F and F_m respectively. Then*

$$\lim_{m \rightarrow \infty} D_m(\lambda) = D(\lambda) \quad \forall \lambda \in \mathbb{R}.$$

Proof. Let \dim_X be the von Neumann dimension associated with the trace τ_X on \mathcal{A}_X . Note that $\dim \mathcal{H} = \dim \ker B + \dim \operatorname{im} B$ for an operator B in a von Neumann algebra of operators acting on a Hilbert space \mathcal{H} , with finite von Neumann dimension \dim . So

$$\begin{aligned} D(\lambda) &= \dim_\tau \ker(A - \lambda) = d - \dim_\tau \operatorname{im}(A - \lambda), \\ D_m(\lambda) &= \dim_{X_m} \ker(A^{(m)} - \lambda) = d - \dim_{X_m} \operatorname{im}(A^{(m)} - \lambda). \end{aligned}$$

As in the proof of Theorem 6.2, let κ be the propagation of the operator A with respect to the word metric $d_{G/H}$ on G/H and let X'_m be the κ -neighbourhood of X_m so that $f \in \mathcal{H}_{X_m}$ implies $Af \in \mathcal{H}_{X'_m}$; equivalently, $P_{X'_m} A|_{\mathcal{H}_{X_m}} = A|_{\mathcal{H}_{X_m}}$ for all m .

The space $\operatorname{im}(A - \lambda)|_{\mathcal{H}_{X_m}}$ is affiliated with $\mathcal{A}_{X'_m}$. Recall the properties (25), (26) and (27) of \dim_X as listed in the proof of Theorem 6.2. Then

$$\begin{aligned} \dim_{X'_m} \operatorname{im}(A^{(m)} - \lambda) &= \dim_{X'_m} P_{X_m}(\operatorname{im}(A - \lambda)|_{\mathcal{H}_{X_m}}) \\ &\leq \dim_{X'_m} \operatorname{im}(A - \lambda)|_{\mathcal{H}_{X_m}} \\ &\leq \dim_\tau \operatorname{im}(A - \lambda), \end{aligned}$$

and

$$\dim_{X_m} \operatorname{im}(A^{(m)} - \lambda) = \frac{\#X'_m}{\#X_m} \dim_{X'_m} \operatorname{im}(A^{(m)} - \lambda).$$

The X_m constitute a Følner exhaustion of G/H and so $\frac{\#X'_m}{\#X_m}$ tends to 1 as m goes to infinity. Taking limits gives

$$\begin{aligned} \liminf_m D_m(\lambda) &= d - \limsup_m \dim_{X_m} \operatorname{im}(A^{(m)} - \lambda) \\ &\geq d - \dim_\tau \operatorname{im}(A - \lambda) = D(\lambda). \end{aligned}$$

Finally, applying Lemma 7.6 gives

$$D(\lambda) \leq \liminf_m D_m(\lambda) \leq \limsup_m D_m(\lambda) \leq D(\lambda).$$

\square

The proof of Theorem 7.1 now follows from Lemmas 7.4 and 7.5, and Theorem 7.7.

The following corollary is an immediate consequence of Theorem 6.6 and Theorem 7.1.

Corollary 7.8 (Generalized IDS). *Let $G = \Gamma$ be the fundamental group of a closed Riemann surface of genus $g > 1$, $G/H = \mathbb{Z}^{2g}$ be the abelianisation of G , in which case the commutator subgroup $H = F$ is a free group. Then the equality between the generalized integrated density of states and the von Neumann spectral density function given in equation (32) holds for every multiplier σ on Γ .*

Corollary 7.9 (Criterion for spectral gaps). *Consider the situation in Theorem 7.1. The interval (λ_1, λ_2) is in a gap in the spectrum of A if and only if*

$$(41) \quad \lim_{m \rightarrow \infty} (F_m(\lambda_2) - F_m(\lambda_1)) = 0.$$

Proof. The interval (λ_1, λ_2) is in a gap in the spectrum of A if and only if $F(\lambda_2) = F(\lambda_1)$. By Theorem 7.1, this is true if and only if

$$\lim_{m \rightarrow \infty} (F_m(\lambda_2) - F_m(\lambda_1)) = F(\lambda_2) - F(\lambda_1) = 0.$$

□

8. THE CLASS \mathcal{K} AND EXTENSIONS WITH CYCLIC KERNEL

In this section we prove the results cited in the earlier sections concerning the class of groups \mathcal{K} . Namely we show that the class \mathcal{K} is closed under taking extensions with cyclic kernel, and that every group in \mathcal{K} has the algebraic eigenvalue property.

Recall that the class \mathcal{K} is the smallest class of groups which contains the free groups and the amenable groups, and is closed under directed unions and under taking extensions with amenable quotients. It can be seen that every group in \mathcal{K} must belong to some \mathcal{K}_α defined inductively as follows.

Definition 8.1. Define the nested classes \mathcal{K}_α , α an ordinal, by

- \mathcal{K}_0 consists of all free groups and all discrete amenable groups,
- $\mathcal{K}_{\alpha+1}$ consists of all extensions of groups in \mathcal{K}_α with amenable quotient, and all directed unions of groups in \mathcal{K}_α ,
- $\mathcal{K}_\beta = \bigcup_{\alpha < \beta} \mathcal{K}_\alpha$ when β is a limit ordinal.

A group is in \mathcal{K} if and only if it is in a class \mathcal{K}_α for some ordinal α .

Recall Theorem 3.7:

Theorem. *Every group in \mathcal{K} has the algebraic eigenvalue property.*

The proof follows closely that of Corollary 4.8 of [9], and relies upon the same key theorem:

Theorem (4.7 of [9]). *Let H have the algebraic eigenvalue property. Let G be a generalized amenable extension of H . Then G has the algebraic eigenvalue property.*

The proof then proceeds by transfinite induction.

Proof of Theorem 3.7. The algebraic eigenvalue property holds for groups in \mathcal{K}_0 by Corollary 4.8 in [9].

Proceeding by transfinite induction, suppose \mathcal{K}_α has the algebraic eigenvalue property for all α less than some given ordinal β .

When $\beta = \alpha + 1$ for some ordinal α , a group G is in \mathcal{K}_β if it is an extension of a group in \mathcal{K}_α with amenable quotient, or is the directed union of groups $G_i \in \mathcal{K}_\alpha$.

Note that $A \in M_d(\overline{\mathbb{Q}}G)$ can be regarded as a matrix A' in $M_d(\overline{\mathbb{Q}}H)$ where H is a finitely generated subgroup of G , generated by the finite support of the $A_{i,j}$ in G . By Proposition 3.1 of [20], the spectral density functions of A and A' coincide. As subgroups of a group with the algebraic eigenvalue property also have the property, it follows that a group has the algebraic eigenvalue property if and only if it holds for all of its finitely generated subgroups. If $G \in \mathcal{K}_{\alpha+1}$ is the directed union of groups $G_i \in \mathcal{K}_\alpha$, it follows that every finitely generated subgroup of G is in some G_i and so has the algebraic eigenvalue property. G therefore has the algebraic eigenvalue property.

Suppose that G is the extension of a group H in \mathcal{K}_α with amenable quotient $R = G/H$, i.e. $H \rightarrow G \xrightarrow{p} R$, then $R = \bigcup_{j \in J} R_j$ is a directed union, where R_j , $j \in J$ are finitely generated and amenable (since amenability is subgroup closed). Consider the extensions $G_j = p^{-1}(R_j)$ of H , with finitely generated amenable quotient: then Theorem 4.12 of [9] applies to show that G_j has the algebraic eigenvalue property.

But $G = \bigcup_{j \in J} G_j$ is the directed union of groups G_j having the algebraic eigenvalue property, so G also has the algebraic eigenvalue property by the argument given in the previous paragraph.

Finally, let β be a limit ordinal, so that $\mathcal{K}_\beta = \bigcup_{\alpha < \beta} \mathcal{K}_\alpha$. If G is in \mathcal{K}_β , then it is in \mathcal{K}_α for some $\alpha < \beta$, and so has the algebraic eigenvalue property by the induction hypothesis.

Therefore groups in \mathcal{K}_β have the algebraic eigenvalue property, and the result follows by induction. \square

We next address Proposition 4.4. Its proof follows from the following lemmas.

Lemma 8.2. *The class \mathcal{K}_α is subgroup-closed for all ordinals α .*

Proof. The classes of free groups and amenable groups are both closed under taking subgroups, so \mathcal{K}_0 is subgroup-closed.

Suppose \mathcal{K}_α is subgroup closed for all $\alpha < \beta$, let $G \in \mathcal{K}_\beta$ and let H be a subgroup of G . If β is a limit ordinal then $G \in \mathcal{K}_\alpha$ for some $\alpha < \beta$, and so $H \in \mathcal{K}_\alpha$, and so in \mathcal{K}_β .

Suppose then that β is not a limit ordinal, with $\beta = \alpha + 1$ for some ordinal α . If G is a directed union of groups G_j in \mathcal{K}_α , then H is a directed union of groups $H \cap G_j$ which are also in \mathcal{K}_α by the induction hypothesis.

G must otherwise be an extension of a group N in \mathcal{K}_α with amenable quotient B . In this case, H is an extension of $N \cap H$ with quotient $H/(N \cap H) \cong NH/N \subset G/N \cong B$, and hence is an extension with amenable quotient of $N \cap H$ which is in \mathcal{K}_α by the hypothesis. H is therefore in \mathcal{K}_β . \square

Lemma 8.3. *Let G be a group whose j -th derived group $G^{(j)}$ is in \mathcal{K}_α . Then $G \in \mathcal{K}_{\alpha+j}$.*

Proof. G is an extension with kernel its derived group G' by the short exact sequence $G' \rightarrow G \rightarrow G/G'$. As the quotient is abelian (and thus amenable), $G' \in \mathcal{K}_\alpha$ implies $G \in \mathcal{K}_{\alpha+1}$, and the result follows inductively. \square

Lemma 8.4. *Let G be an extension of $H \in \mathcal{K}_0$ with cyclic kernel. Then $G \in \mathcal{K}_2$.*

Proof. Let A be the cyclic group, so that the sequence

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

is exact.

If H is amenable, then G is an amenable group, since A is amenable and the class of amenable groups is closed under taking extensions.

Suppose instead that H is free. As free groups have trivial second cohomology, G must be the semi-direct product $A \rtimes_{\phi} H$ with $\phi : H \rightarrow \text{Aut } A$.

As before, denote the derived groups of G and H by G' and H' respectively. Then the following diagram commutes with exact rows, and where the vertical homomorphisms are just inclusions.

$$\begin{array}{ccccccc} 1 & \longrightarrow & A \cap G' & \longrightarrow & G' & \xrightarrow{\pi|_{G'}} & H' \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & H \longrightarrow 1 \end{array}$$

G' is a semidirect product $A \cap G' \rtimes_{\phi|_{H'}} H'$, and $\phi|_{H'}$ must have image in $(\text{Aut } A)'$.

As A is cyclic, it has abelian automorphism group, and so $(\text{Aut } A)'$ is trivial. This in turn implies that $\phi|_{H'}$ is trivial, and that $G' \cong (A \cap G') \times H'$. It follows then that $G'' \cong H''$, which is free and thus in \mathcal{K}_0 . Therefore G is in \mathcal{K}_2 . \square

Proof of Proposition 4.4. Proceeding by transfinite induction, suppose that for a given ordinal β , $H \in \mathcal{K}_{\alpha}$ implies that every extension of H with cyclic kernel is in \mathcal{K} for all $\alpha < \beta$.

Let $H \in \mathcal{K}_{\beta}$, and G an extension of H with

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

exact and A cyclic. Choose $\alpha < \beta$ such that $H \in \mathcal{K}_{\alpha+1}$; when β is not a limit ordinal, one can just let $\alpha = \beta - 1$. There are two possibilities: either the group H is a directed union of groups H_j in \mathcal{K}_{α} , $j \in J$; or H is an extension of a group \bar{H} in \mathcal{K}_{α} .

In the first case, $G = \cup_{j \in J} G_j$ with $G_j = \pi^{-1}(H_j)$. Each G_j is an extension of $H_j \in \mathcal{K}_{\alpha}$ with cyclic kernel, and so $G_j \in \mathcal{K}$ by the induction hypothesis. Therefore $G \in \mathcal{K}$.

In the second case, let $B = H/\bar{H}$ be the quotient of the extension, and denote the surjection $H \rightarrow B$ by η . Let $\bar{G} = \pi^{-1}(\bar{H})$. Then $\bar{G} = \ker \eta\pi$ and we get the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & \bar{G} & \xrightarrow{\pi|_{\bar{G}}} & \bar{H} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & H \longrightarrow 1 \\ & & & & \downarrow \eta\pi & & \downarrow \eta \\ & & & & B & \xlongequal{\quad} & B \end{array}$$

G is therefore an extension of \bar{G} with amenable quotient B , while \bar{G} is an extension of $\bar{H} \in \mathcal{K}_{\alpha}$ with cyclic kernel A . So $\bar{G} \in \mathcal{K}$ by the induction hypothesis, and so $G \in \mathcal{K}$.

The case for $\beta = 1$ holds by virtue of Lemma 8.4, and so by induction, $H \in \mathcal{K}$ implies that any extension with cyclic kernel of H is in \mathcal{K} . \square

Remark 8.5. Linnell's class \mathcal{C} is also closed under taking extensions with cyclic kernel, by the same argument. \mathcal{C} can be written as the union of classes \mathcal{C}_α for ordinals α :

- \mathcal{C}_0 is the class of free groups.
- $\mathcal{C}_{\alpha+1}$ is the class of groups which are extensions of groups in \mathcal{C}_α with elementary amenable quotient, or are directed unions of groups in \mathcal{C}_α .
- $\mathcal{C}_\beta = \bigcup_{\alpha < \beta} \mathcal{C}_\alpha$ when β is a limit ordinal.

As the abelian groups are all elementary amenable, the above procedure for \mathcal{K} can be applied directly to \mathcal{C} .

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